IIT-JEE / ENGINEERING

MATHS

FORMULA

Three Dimensional Geometry

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IMPORTANT DEFINATIONS, FORMULAE AND METHODS

1. **Direction cosines of a line**

(i) Direction cosines of a line are the cosines of the angles made by the line X-axis,

Y-axis and Z-axis.

(ii) Direction cosines of X-axis, Y-axis and Z-axis are respectively 1, 0, 0; 0, 1, 0 and 0, 0, 1.

(iii) If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.

2. **Direction ratios of a line**

(i) These are the numbers say a, b, c which are proportional to the direction cosines

l, m, n i.e. $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

(ii) $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$, n $= \frac{c}{\sqrt{a^2+b^2+c^2}}$

(iii) Let AB be a line joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Then the direction ratios of the line AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and its direction cosines are $\frac{x_2 - x_1}{|AB|}, \frac{y_2 - y_1}{|AB|}, \frac{z_2 - z_1}{|AB|}$, where $|AB| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.

(iv) The direction ratios of vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ are a, b, c.

3. **Angle between two lines**

(i) If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines and θ is the angle

between them, then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

(ii) If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the angle

between them, then $\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

(iii) For perpendicular lines, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

(iv) For parallel lines, $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

4. Equation of a line passing through a given point and parallel to a given vector

(i) The vector form of a line passing through a point $A(\vec{a})$ and parallel to given vector $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a scalar.

(ii) Cartesian equation of a line passing through $A(x_1, y_1, z_1)$ and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$

5. Equation of a line passing through two given points

(i) The vector form of a line passing through two points $A(\vec{a})$ and $B(\vec{b})$ is given by

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

(ii) Cartesian equation of a line passing through points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda$

6. Angle between two lines (in vector form)

If θ is an angle between $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, then

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1|\,|\vec{b}_2|}$$

7. Skew lines

(i) Two lines in a space which are neither parallel nor intersecting are called skew lines.

(ii) Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.

8. Shortest distance between two skew lines

(i) Shortest distance between two skew lines is the perpendicular to both the lines.

(ii) The shortest distance between the skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by,

$$\text{S.D.} = \frac{|(\vec{b}_1 X \vec{b}_2).(\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 X \vec{b}_2|}$$

(iii) Shortest distance between the skew lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=\lambda$ and

$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}=\mu$ is

$$S.D.=\frac{1}{\sqrt{\sum(a_1b_2-b_1a_2)^2}}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \end{vmatrix}$$

(iv)Distance between the parallel lines $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is given by

$$\left|\frac{\vec{b}\ X\ (\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|.$$

9. Condition for two given lines to intersect

Condition for two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=\lambda$ and

$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}=\mu$ to

(i) Intersect is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \end{vmatrix}=0$

(ii) do not intersect is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\neq 0$

10. Equation of a plane in the normal form

(i) In the vector form, equation of a plane which is at a distance 'd' from the origin and $\hat{n}$ is a unit vector normal to the given plane, directed from the origin to the plane, is

$$\vec{r}\ .\hat{n}=d$$

(ii) Equation of a plane which is at a distance'd' from the origin and l, m , n are the direction cosines of the normal to the plane is

$$lx+my+nz=d$$

(iii) The equation of a plane passing through a point $A(\vec{a})$ and perpendicular to the given vector $\vec{n}$ is given by $\vec{r}.\vec{n}=\vec{a}\ .\vec{n}$

(iv) The equation of a plane passing through a point $A(x_1, y_1, z_1)$ and perpendicular to the given line with direction ratios a, b, c is given by

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

11. General equation of a plane

General equation of a plane having a, b, c as direction ratios of the normal to the plane is $ax+by+cz=d$.

12. Method to reduce general form of equation to the normal form

Let the general equation of the plane is ax + by + cz + d = 0

$\Rightarrow \quad -ax-by-cz=d$

Dividing both sides by $\sqrt{(-a)^2+(-b)^2+(-c)^2}$

We get, $\dfrac{-ax}{\sqrt{a^2+b^2+c^2}}+\dfrac{-by}{\sqrt{a^2+b^2+c^2}}+\dfrac{-cz}{\sqrt{a^2+b^2+c^2}}=\dfrac{d}{\sqrt{a^2+b^2+c^2}}$

$\Rightarrow \quad lx+my+nz=p$

13. Equation of a plane parallel to a given plane

(i) The equation of a plane parallel to the plane $\vec{r}.\vec{n}=d$ is given by $\vec{r}.\vec{n}=K$, where the constant K is determined by a given condition.

(ii) The equation of a plane parallel to the plane $ax+by+cz+d=0$ is given by $ax+by+cz+K=0$, where K is determined by a given condition.

14. Equation of a plane through the intersection of two planes

(i) Vector equation of a plane that passes through the intersection of two planes $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$ given by $\vec{r}.(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$, where λ is any parameter.

(ii) Cartesian equation of a plane that passes through the intersection of two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given by $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d)=0$, where λ is any parameter.

15. Equation of a plane passing through three given points

(i) The vector equation of a plane passing through three non collinear points $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ is given by $(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a})\text{ X }(\vec{c}-\vec{a})\right]=0$

(ii) Cartesian equation of a plane passing through three non collinear points

$A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

16. Equation of a plane passing through a given point and parallel to two given lines

(i) The vector equation of a plane passing through a given point $A(\vec{a})$ and parallel to the given vectors $\vec{b}$ and $\vec{c}$ is $(\vec{r}-\vec{a}).(\vec{b}\ X\ \vec{c}) = 0$

(ii) Cartesian equation of a plane passing through a given point $A(x_1, y_1, z_1)$ and parallel to two given lines having direction ratios b_1, b_2, b_3 and c_1, c_2, c_3 is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

17. Angle between two planes :

(i) The angle between two given planes is the angle between their normals.

(ii) In the vector form, if θ is the angle between the two planes $\vec{r}\,.\,\vec{n}_1 = d_1$ and $\vec{r}\,.\,\vec{n}_2 = d_2$, then $\cos\theta = \dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

(iii) In the Cartesian form, if θ is the angle between the two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

18. Angle between a line and a plane

(i) The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

(ii) In vector form, if θ is the angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}\,.\,\vec{n} = d$, then $\sin\theta = \dfrac{\vec{b}\,.\,\vec{n}}{|\vec{b}||\vec{n}|}$

(iii) In Cartesian form, if θ is the angle between the line $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and the plane $a_2x+b_2y+c_2z+d=0$, then

$$\sin\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

19. Coplanarity of two lines

(i) In vector form, two planes $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are coplanar if $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\ X\ \vec{b}_2)=0$.

(ii) In Cartesian form, two panes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are coplanar if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

20. Coplanarity of two lines

(i) In vector form, two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are coplanar if $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\ X\ \vec{b}_2)=0$.

(ii) In Cartesian form, two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

21. Length of perpendicular from a point to a plane

(i) In vector form, length of perpendicular drawn point $A(\vec{a})$ to the plane $\vec{r}.\vec{n}=d$ is given by $\frac{|\vec{a}\ .\vec{n}-d|}{|\vec{n}|}$.

(ii) In Cartesian form, the length of the perpendicular drawn from $A(x_1,y_1,z_1)$ to the plane $ax+by+cz+d=0$ is given by $\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

22. Equation of a plane in the intercept form

Equation of a plane that makes intercepts of lengths a, b, c with the X-axis, Y-axis and Z-axis respectively is given by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.