IIT-JEE / ENGINEERING

MATHS

FORMULA

Continuity and Differentiability

By

BHARAT BHUSHAN @ B. K. NAL

Assistant Professor (Computer Science)

Director, BSTI, Kokar

&

SUPRIYA BHARATI

Assistant Professor (Computer Science)

Asst. Director, BSTI, Kokar

Buddha Science & Technical Institute

Kokar, Ranchi-834001, Jharkhand, India

www.bharatsir.com

IMPORTANT DEFINATIONS, FORMULAE AND METHODS

1. **Continuity at a point :** A function f is said to be continuous at x = a if $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$. In other words, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$, then f is continuous at x = a.

2. **Discontinuity and its types :** If the function f is not continuous at x = a, then f is said to be discontinuous at x = a. There are three types of discontinuity :

   (a) Removable Discontinuity : If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$ then f has removable discontinuity at x = a. Such type of discontinuity can be removed by redefining the function.

   (b) Discontinuity of First Kind : If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then f has discontinuity of first kind.

   (c) Discontinuity of Second Kind : If atleast one of the limits, $\lim_{x \to a^-} f(x)$ *or* $\lim_{x \to a^+} f(x)$, does not exists finitely, at x = a then f has discontinuity of second kind.

3. **Continuous Function :** If the function f is continuous at every point of its domain, then f is said to be a continuous function. There are some continuous functions. For examples :

   (a) A constant function is everywhere continuous.

   (b) A polynomial function is everywhere continuous.

   (c) A rational function is continuous in its domain.

   (d) Logarithmic function is continuous on $(0, \infty)$.

   (e) Exponential function is everywhere continuous.

   (f) Modulus function is everywhere continuous.

   (g) Trigonometric functions are continuous in their domains.

4. **Important Note :** If f and g are any two continuous functions, then f + g, f - g, fg, f/g (provided $g(x) \neq 0$) are also continuous function.

5. **Continuity on an Interval :** A function f is said to be continuous on an interval [a, b] if it is continuous at every points in [a, b] including the end point a and b.

Continuity of f at a means. $\lim_{x\to a^+} f(x) = f(a)$ and continuity of f at b means $\lim_{x\to b^-} f(x) = f(b)$

6. **Method to examine the Continuity :** The continuity of the function f can be examined as under :

I. Let $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$ at x = 2 when $x \to 2^-$ or $x \to 2^+$, $f(x)$ remains same, so find $\lim_{x\to 2} f(x)$ and use definition of continuity. It means, if $\lim_{x\to 2} f(x) = f(2)$ then f is continuous at x = 2 otherwise discontinuous.

II. In case of modulus function, greatest integer function exponential function (e^x) or function like $f(x) = \begin{cases} x+2, & \text{if } x < 1 \\ 3, & \text{if } x = 1 \\ 2x-1, & \text{if } x > 1 \end{cases}$ at x = 1, $f(x)$ is different for LHL and RHL, so find LHL and RHL separately and then check whether LHL = RHL = f(a) or not.

7. **Differentiability :** A function f is said to be differentiable at x = a if $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists finitely i.e., if $\lim_{x\to a^-} \dfrac{f(x)-f(a)}{x-a} = \lim_{x\to a^+} \dfrac{f(x)-f(a)}{x-a}$, **if a function f is differential at any point, then it must be continuous at that point. But converse is not necessarily true.**

8. **Important formulae :** If $y = f(x)$, then derivative of y w.r.t. x is written as $\dfrac{dy}{dx}$ or y_1 or $f'(x)$. We can also use $\dfrac{d}{dx} \equiv$ D.

The second order derivative of $y = f(x)$, is written by $\dfrac{d^2y}{dx^2}\left[\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)\right]$ or y_2 or $f''(x)$

i. $D(x)^n = nx^{n-1}$

ii. D (Constant) = 0

iii. $D(\sqrt{f(x)}) = \dfrac{1}{2\sqrt{f(x)}} X f'(x)$

iv. $D\left[\dfrac{1}{f(x)}\right] = \dfrac{-1}{(f(x))^2} X f'(x)$

v. $D[f(x)]^n = n[f(x)]^{n-1} X f'(x)$

vi. $D(\sin x) = \cos x$

vii. $D(\cos x) = -\sin x$

viii. $D(\tan x) = \sec^2 x$

ix. $D(\cot x) = -\operatorname{cosec}^2 x$

x. $D(\sec x) = \sec x \ \tan x$

xi. $D(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

xii. $D[\sin[f(x)]] = \cos[f(x)] \text{ X } f'(x)$

xiii. $D(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

xiv. $D(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

xv. $D(\tan^{-1} x) = \dfrac{1}{1+x^2}$

xvi. $D(\cot^{-1} x) = \dfrac{-1}{1+x^2}$

xvii. $D(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

xviii. $D(\operatorname{cosec}^{-1} x) = \dfrac{-1}{x\sqrt{x^2-1}}$

xix. $D(e^x) = e^x$

xx. $D[e^{f(x)}] = e^{f(x)} \text{ X } f'(x)$

xxi. $D(a^x) = a^x \log a, (a > 0)$

xxii. $D(\log x) = \dfrac{1}{x}$

xxiii. $D[\log f(x)] = \dfrac{1}{f(x)} X \ f'(x)$

xxiv. $D[f(x) \pm g(x)] = D[f(x)] \pm D[g(x)]$

xxv. $D[C\,f(x)] = C[D\,f(x)]$, C = constant

xxvi. $D[f(x) \text{ X } g(x)] = f(x)\dfrac{d}{dx}[g(x)] + g(x)\dfrac{d}{dx}[f(x)]$

xxvii. $D\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$

xxviii. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

xxix. $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$

xxx. $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

9. **Chain Rule :** If y is a function of x and x is a function of u, then y is a composite function of u. By chain rule $\frac{dy}{du}=\frac{dy}{dx}\ X\ \frac{dx}{du}$

10. **Derivatives of Implicit Function :** A function of the type $y=f(x)$ is called an explicit function and a function of the type $f(x,y)=c$, where c is any constant is called an implicit function. For example, $x^2+y^2+xy=2$ is an implicit function. Let us differentiate it w.r.t. x, we have

$$2x+2y\frac{dy}{dx}+\left[x\frac{dy}{dx}+y \text{ x } 1\right]=0$$

$$\Rightarrow \quad (2y+x)\frac{dy}{dx}=-(2x+y)$$

$$\therefore \quad \frac{dy}{dx}=-\left[\frac{2x+y}{2y+x}\right]$$

NOTE : If possible, first express the implicit function as an explicit function, then differentiate.

11. **Derivatives of parametric functions :** If $x=f(\theta)$ and $y=f(\theta)$, then such functions are called parametric functions. Here θ is the parameter. Using chain rule

$$\frac{dy}{dx}=\frac{dy}{d\theta}\ X\ \frac{d\theta}{dx}$$

12. **Logarithmic differentiation :** If we have the function of the types $y=[f(x)]^{f(x)},[f(y)]^{f(x)}=[f(x)]^{f(y)}$ etc., then first take log. Both sides, then differentiate. For example, if $y=(x)^{\log x}$. Taking log both sides $\log y=(\log x)(\log x)=(\log x)^2$. Now differentiate it w.r.t. x and find $\frac{dy}{dx}$.

13. **Derivative of a function w.r.t. another function :** To differentiate a function $f(x)$ w.r.t. another function $g(x)$, let us suppose that $A=f(x)$ and $B=g(x)$, then using chain rule, we have $\frac{dA}{dB}=\frac{dA}{dx}\ X\ \frac{dx}{dB}$

14. Rolle's Throrem : If $f: [a, b] \to R$ is continuous on [a, b] and differentiable on (a, b) such that $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

15. Lagrange's Mean Value (LMV) Theorem : If $f:[a,b] \to R$ is continuous on [a, b] and differentiable on (a, b). Then there exists atleast one $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Remarks :

I. If $f(x)$ is of the type $f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$, then $f(x)$ remains same for L.H.L. and R.H.L., so to check continuity at x = 0 here use the following : $\lim_{x \to 0} f(x) = f(0)$

II. If the function is of the type $f(x) = |x|$, $f(x) = \begin{cases} 2x - 1, & x < 1 \\ 1, & x = 1 \\ 2x + 1, & x > 1 \end{cases}$ etc, then $f(x)$ is different for LHL and RHL, so to examine continuity at any point a, use the following $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

Note : if any mistake on this, kindly inform on the mail id :

bknal1207@gmail.com