IIT-JEE / ENGINEERING

MATHS

FORMULA

INTEGRALS

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IMPORTANT DEFINATIONS, FORMULAE AND METHODS

1. **Integration :** The inverse process of differentiation is called integration. In Integral Calculus there are two types of integrals, indefinite integrals and definite integrals.

2. **INDEFINITE INTEGRALS :** The indefinite integrals of $f(x)$ is $\int f(x)dx = F(x) + C$, where $\frac{d}{dx}[F(x)] = f(x)$ and C is called constant of integration. Since C is here arbitrary, therefore these integrals are called indefinite integrals.

3. **Properties of Indefinite Integrals :**

   (I) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

   (II) $\int Kf(x)dx = K\int f(x)dx$, where K is any real number.

4. **Formulae**

   (I) $\int x^n dx = \frac{x^{n+1}}{n+1} + C \; (n \neq -1)$

   (II) $\int 1.\, dx = x + C$

   (III) $\int \frac{1}{x} dx = \log x + C$

   (IV) $\int e^x dx = e^x + C$

   (V) $\int a^x dx = \frac{a^x}{\log x} + C$

   (VI) $\int \sin x dx = -\cos x + C$

   (VII) $\int \cos x dx = \sin x + C$

   (VIII) $\int \tan x dx = \log(\sec x) + C$ or -log $|\cos x| + C$

   (IX) $\int \cot x dx = \log(\sin x) + C$

   (X) $\int \sec x dx = \log(\sec x + \tan x) + C$

   (XI) $\int \operatorname{cosec} x dx = \log(\operatorname{cosec} x - \cot x) + C$

   (XII) $\int \sec x \tan x dx = \sec x + C$

   (XIII) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$

(XVI) $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$

(XVII) $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + C$

(XVIII) $\int \sec^2 x dx = \tan x + C$

(XIX) $\int \cos ec^2 x dx = -\cot x + C$

5. NOTE : The formula $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$ is only valid for algebraic polynomials, provided the base of the exponent must be linear.

$$\int (x^2-1)^2 dx = \frac{(x^2-1)^3}{3X\,2x} + C$$

is the wrong method. The correct method is

$$\int (x^2-1)^2 dx = \int (x^4 - 2x^2 + 1)dx = \frac{x^5}{5} - \frac{2x^3}{3} + x + C$$

6. Methods of Integration :

To find the integrals, the following methods are used :

(I) Integration by decomposition.

(II) Integration by substitution, including standard integrals.

(III) Integration by partial fractions.

(IV) Integration by parts.

7. Integration by decomposition :

In this method, the integrand is decomposed. See some examples :

(I) $\int (x^2 + 1 + e^x + \sin x)dx = \int x^2 dx + \int 1.\,\mathrm{dx} + \int e^x dx + \int \sin x dx$

(II) $\int \sin^2 x dx = \frac{1}{2}\int (1 - \cos 2x) = \frac{1}{2}\int 1.\,dx - \frac{1}{2}\int \cos 2x dx$

(III) $\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(x+4)-4}{\sqrt{x+4}} = \int \left(\sqrt{x+4} - 4\ X\ \frac{1}{\sqrt{x+4}}\right)dx = \int \sqrt{x+4}dx - 4\int \frac{1}{\sqrt{x+4}}dx$

(IV) $\int \frac{1}{\sin^2 x \cos^2 x}dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx$

$= \int (\sec^2 x + \cos ec^2 x)dx = \int \sec^2 x dx + \int \cos ec^2 x dx$ etc.

8. Integration by Substitution including standard integrals :

The method in which we change the variable to some other variable is called the method of substitution.

When the integrand is of the type $f(x)\ X\ f'(x)$ or $\frac{f'(x)}{f(x)}$ then we put $f(x)=t$ and $f'(x)dx = dt$.

For example $\quad I=\int\frac{(\log x)^2}{x}dx$

$$=\int(\log x)^2\ X\ \frac{1}{x}=\int[f(x)]^2 X\ f'(x)$$

So let $\quad \log x = t \Rightarrow \frac{1}{x}dx = dt$

$$I=\int t^2 dt=\frac{t^3}{3}+C=\frac{(\log x)^3}{3}+C$$

Standard Integrals : The integrals of the types $\int\frac{dx}{ax^2+bx+c}$, $\int\frac{dx}{\sqrt{ax^2+bx+c}}$, $\int\sqrt{ax^2+bx+c}\ dx$, $\int\frac{px+q}{ax^2+bx+c}dx$, $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$, $\int(px+q)\sqrt{ax^2+bx+c}$ dx are standard integrals.

(a) To find the integrals $\int\frac{dx}{ax^2+bx+c}$, $\int\frac{dx}{\sqrt{ax^2+bx+c}}$, $\int\sqrt{ax^2+bx+c}\ dx$, express ax^2+bx+c as per square and use the following formulae:

(I) $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(II) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(III) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(IV) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C$

(V) $\int\frac{dx}{\sqrt{x^2+a^2}}=\log(x+\sqrt{x^2+a^2})+C$

(VI) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log(x+\sqrt{x^2-a^2})+C$

(VII) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(VIII) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2 + a^2}) + C$

(IX) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2 - a^2}) + C$

For example:

$$\int \frac{dx}{\sqrt{x^2 - 4x - 3}} = \int \frac{dx}{\sqrt{x^2 - 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 - 3}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 4x + 4 - 7}} = \int \frac{dx}{\sqrt{(x-2)^2 - (\sqrt{7})^2}}$$

$$= \log\left[(x-2) + \sqrt{(x-2)^2 - (\sqrt{7})^2}\right] + C$$ [Using VI]

$$= \log\left[(x-2) + \sqrt{x^2 - 4x - 3}\right] + C$$

(b) To find the integrals

$$\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int (px+q)\sqrt{ax^2+bx+c}\ dx$$

Express $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B$

Now by equating the coefficients of like terms both sides, find A and B. Using method of substitution and formulae of standard integrals I to IX, we can find the given integrals.

For example : $I = \int \frac{3x+2}{x^2 + 2x + 3}dx$

Let $3x + 2 = A(2x + 2) + B$

$\Rightarrow \quad 3 = 2A \Rightarrow A = \frac{3}{2}$

And $2 = 2A + B = 3 + B$

$\Rightarrow \quad B = -1$

$\therefore \quad 3x+2=\frac{3}{2}(2x+2)-1$

Or we can use direct method

$$3x+2=3\left(x+\frac{2}{3}\right)=\frac{3}{2}\left(2x+\frac{4}{3}\right)=\frac{3}{2}\left(2x+2-2+\frac{4}{3}\right)$$

$$=\frac{3}{2}\left[(2x+2)-\frac{2}{3}\right]=\frac{3}{2}(2x+2)-1$$

$$\therefore \quad I=\frac{3}{2}\int\frac{2x+2}{(x^2+2x+3)}dx-\int\frac{dx}{(x^2+2x+3)}=\frac{3}{2}I_1-I_2$$

Now find I_1 using method of substitution and I_2 using steps involved in section (a).

9. Integration by Partial fraction : The integrals of the types $\int\frac{dx}{(x+1)(x+2)}$, $\int\frac{xdx}{(x+2)^2(x-3)}$, $\int\frac{x^2}{(x^2+4)(x-5)}dx$ etc, can be evaluated by using method of partial fraction. If the (degree of num.) $\geq$(degree of den.), then use long division and express the given fraction as proper fraction. The proper fraction can be expressed as the sum of partial fractions of the following types :

(I) $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$

(II) $\frac{px+q}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$

(III) $\frac{px+q}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

where x^2+bx+c cannot be factorized further.

10. Integration by Parts : The integrals of the type $\int f(x)g(x)dx$ can be found using integration by parts as follows :

$$\int \underset{I}{f(x)}\ X\ \underset{II}{g(x)}dx=f(x)\left[\int g(x)dx\right]-\int\left[\int g(x)dx\right]X\ f'(x)dx$$

Choose I and II function according to rule ILATE.

NOTE : For the integral $\int e^x\left[f(x)+f'(x)\right]dx$, split it into two integrals as $\int e^x f(x)dx+\int e^x f'(x)dx$ and then apply integration by parts in first integral only.

11. Definite Integrals : We denote the definite integral by $\int_a^b f(x)dx$. Geometrically, $\int_a^b f(x)dx$ is the area bounded by the curves $y = f(x)$, x = a, x = b and the X-axis.

There is no need of constant of integration since the definite integral has a unique value. By fundamental theorem of calculus

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \text{ where } \frac{d}{dx}[F(x)] = f(x)$$

12. Properties of definite integral : The properties of definite integral are very useful in evaluating the definite integral more easily. There are some properties of definite integral given below :

P1 : $\int_{-a}^b f(x)dx = \int_a^b f(t)dt$

P2: $\int_a^b f(x)dx = -\int_a^b f(x)dx$

P3: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

P4: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

P5: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

P6: $$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

P7: $$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

P8: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \text{ where } a < c < b$

13. Definite Integral as the limit of a sum :

Let $I = \int_a^b f(x)dx$. In this method the area bounded by the curves $y = f(x)$, x = a, x = b and the X- axis is divided into small stripes of equal width h. Let the interval [a, b] is divided into n equals sub intervals, then $h = \frac{b-a}{n} \Rightarrow nh = b-a$ and by definition of limit of a sum

$$I = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + ... + f[a+(n-1)]h \right]$$

Note : if any mistake on this, kindly inform on the mail id :

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